**6.5.5 Solution of the General Eigenvalue Problem**

**Concept**: Consider the general eigenvalue problem (6.134):



Assume that the eigenvalue  with the smallest absolute value is to be determined by vector iteration. The start vector  of the iteration is chosen freely. In step s of the iteration, the vector  is used in equation (6.135) to compute an auxiliary vector which is then normalised with respect to **B** in (6.136) to yield the iterated vector  The iteration converges to the eigenstate 



**Convergence to the eigenvector**: The start vector of the iteration is a linear combination  of the eigenvectors. It is modified as follows by the iteration, since none of the eigenvalues equals null:



The product  is eliminated between the last two equations to yield:



Since  is the smallest eigenvalue, the product  will tend to zero as the power s increases, unless there are multiple eigenvalues with absolute value  The term with index k=1 is therefore taken out of the summations in (6.137):



As the power s increases, the root in (6.138) tends towards c1. If c1 differs from null, the limit value of the iterated vector is the eigenvalue 



**Rate of convergence to the eigenvector**: The rate of convergence of the eigenvector  is defined as the limit of the ratio of the **B**-quadratic Form of the error in the iterated vector for steps s and s-1:



Substitution of (6.137) into (6.140) leads to:



Substitution of (6.141) into (6.140) shows that the rate of convergence of the iteration for the eigenvector equals the ratio of the two eigenvalues with the smallest absolute value:



**Special eigenvalues**: All eigenvalues of (6.134) differ from zero. If one of the eigenvalues were zero, the determinant of **A** would be zero. This contradicts the assumption that **A** is regular. If the smallest eigenvalue is multiple, equation (6.138) is replaced by:



The iterated vector now converges to



The real values  andare arbitrary. Let the limit value in (6.144) be **w**. The second eigenvector corresponding to  is determined by choosing a start vector which is **B**-orthogonal to **w**. This is analogous to the determination of the neighbouring eigenstates.

If there are two eigenvalues of equal absolute value but opposite sign, they are orthogonal and equation (6.138) is replaced by:



The iterated vector now converges to



The iterated vector oscillates between the two values on the right hand side of (6.146). Successive limit values yield the two eigenvectors. Their Raleigh quotients yield the corresponding eigenvalues.

**Convergence of the Rayleigh quotient to the eigenvalue:** The eigenvalue for an iterated vector is estimated by means of the Rayleigh quotient 



Expression (6.137) for the iterated vector is substituted into (6.147) in order to obtain an expression for  in terms of the eigenvectors :



As the power s increases, the sums go to null since the ratio || is less than 1.0. The limit value of  is therefore the eigenvalue :



**Rate of convergence of the Rayleigh quotient to the eigenvalue**: The rate of convergence to the Rayleigh quotient is defined as the limit of the ratio of the difference between the Rayleigh quotient and the eigenvector in steps s and s-1 of the iteration:



Expression (6.137) for the iterated vector is substituted into (6.150) in order to obtain an expression for  in terms of the eigenvectors:



An analogous formula is derived for 



Substitution of (6.151) and (6.152) into (6.150) shows that the rate of convergence to the eigenvalue equals the square of the ratio of the eigenvalues with the smallest absolute value:



**Neighbouring eigenstates**: Assume that the eigenstate of (A.1) with the smallest absolute value of the eigenvalue is known. The eigenstate with the second smallest absolute value of the eigenvalue is to be determined by vector iteration. This is achieved by making the start vector  **B**-orthogonal to  Let  be an arbitrary vector which is not **B**-orthogonal to and let the component of **v** which is **B**-orthogonal to be 



The first step of the iteration now yields:

Sincedoes not contain the eigenvector, the iteration will converge to the eigen-vector corresponding to the remaining eigenvalue with the smallest absolute value. Due to the round-off in the numerical solution, the iterated vectormust periodically be made **B**-orthogonal to the eigenvector  In order to be on the safe side, this can be done after each cycle of iteration. Additional neighbouring eigen-states can be determined in analogous fashion:



**Algorithm for the Vector Iteration**: The iteration rules (6.135) and (6.136) are not implemented directly in the algorithm. The repetition of computations is avoided by introducing the following auxiliary vectors:



The algorithm for the computation of the eigenvalue with the smallest absolute value of the general eigenvalue problem (6.134) consists of the following steps:



For the determination of the second eigenstate, an initial value  is chosen for cycle 1. The **B**-orthogonalisation of is replaced by corresponding operations on 



Due to round-off, the value of  computed in cycle s-1 is no longer **B**-orthogonal to  and is therefore denoted by  The **B**-orthogonalisation of  is replaced by corresponding operations on 



Step 7 is extended to include the **B**-orthogonalisation of according to (6.166). If more than two eigenstates are determined, the **B**-orthogonalisation is extended to all eigenvectors that have been determined.

For the computation of limit points, it is usually sufficient to determine the smallest eigenvalue since this determines the limit state. The determination of more than one eigenvalue is only of interest if the point is a bifurcation point and the bifurcation paths are to be determined.